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Writing certain commutators as products of cubes in free groups ☆

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Abstract

Let $\mathrm{Cu}(\gamma)$ be the minimal number of cubes required to express an element γ of a free group F. We establish a method for showing that certain equations do not have solutions in free groups. Using it, we find $\mathrm{Cu}(\gamma)$ for certain elements of the derived subgroup of F. If $W = F \wr C_\infty$ is the wreath product of F by the infinite cyclic group, we also show that every element of W' is a product of at most one commutator and three cubes in W. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

It is known that conjugate elements commute in a group of exponent 3. In [2], Lyndon wrote $[x^y, y]$ as a product of four cubes:

$$[x^y,x] = (y^{-1}x^{-1}y^2)^3 y^{-3}(yx)^3(x^{-1}y^{-1}x^{-1}y^{-1}xyx)^3.$$

This is not optimal. We show that $[x^y,x]$ can be expressed as a product of 3 cubes, but not fewer than 3. We use the following notation:

If γ is an element of a group G, we write $\gamma \in \mathrm{Cu}_n = \mathrm{Cu}_n(G)$ to mean that γ can be written as a product of n cubes in G, and $\mathrm{Cu}(\gamma) = n$ denotes that $\gamma \in \mathrm{Cu}_n$ but $\gamma \notin \mathrm{Cu}_{n-1}$. We call $\mathrm{Cu}(\gamma)$ the cube length of γ. Finally, G^3 is the fully invariant subgroup of G. generated by the cubes of the elements of G.

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2. Main results

Let F be a free group and x, y be two distinct elements of a free generating set. Then $[x^y, x] = [x, y, x]$. We present the following easily verified expression of $[x, y, x]$ as a product of three cubes;

$$[x, y, x] = ((x^{-1}y)^y)^3(y^{-2}x)(y^x)^3. \tag{1}$$

Hence, $\mathrm{Cu}([x, y, x]) \leqslant 3$. To show that 3 is the minimal number, we consider equation $[x, y, x] = A^3B^3$ in F. We reduce the problem to the rank two case, we may assume $F = F(x, y)$ be the free group of rank 2 generated by x, y, and we prove the following results:

Theorem 1. Let $F(x, y)$ *be the free group of rank* 2 *freely generated by* x, y, *the equation* $[x, y, x]^n = A^3B^3$ *has a solution in* F *if and only if* $3|n$. *In particular*, *neither* $[x, y, x] = A^3B^3$ *nor* $[x, y, x]^2 = A^3B^3$ *has a solution.*

To prove this, we need the following lemma. Let F_n denote the nth term of the lower central series of F.

Lemma 1. *Let* $F = F(x, y)$ *be a free group, and* $\gamma \in \mathrm{Cu}_2(F)$. *If* $\gamma \in F_n$ *for some* n *then the image of* γ *in* F_n/F_{n+1} *belongs to* $\mathrm{Cu}_1(F_n/F_{n+1})$ *and the image of* γ *in* F_n/F_{n+2} *belongs to* $\mathrm{Cu}_2(F_n/F_{n+1})$.

We prove following theorem regarding the wreath product of groups.

Theorem 2. *Let* $W = F \wr C_\infty$ *be the wreath product of a free group F and the infinite cyclic group* C_∞. *Then every element of* W' *is a product of at most one commutator and three cubes in W. More precisely let B be the base group of W and let t generate* C_∞. *For any* $w \in W'$ *there is* $a \in B$ *and* $c_1, c_2, c_3 \in W$ *such that*

$$w = [a, t]c_1^3c_2^3c_3^3.$$

3. Proofs

Proof of Lemma 1. If $n = 1, \gamma \in F_1$ hence $\gamma = u^3v^3$ in F. Since $\gamma F' = u^3v^3F' = (uv)^3F', \gamma = (uv)^3$ modulo F'. When $n > 1, \gamma \in F'$, and $(uv)^3 = 1 \,(\mathrm{mod}\, F')$. Since F/F' is torsion free $uv = f \in F'$. Hence $\gamma = u^3v^3 = u^3(u^{-1}f)^3 = u^2fu^{-1}fu^{-1}f = [u^{-1}, f^{-1}]^{u^{-1}}[u^{-1}, f^{-2}]f^3$. But then $\gamma \equiv f^3 \,(\mathrm{mod}\, F_3)$.

Also, we have

$$\begin{aligned}\gamma &\equiv [u^{-1}, f^{-1}][u^{-1}, f^{-1}, u^{-1}][u^{-1}, f^{-1}]^2[u^{-1}, f^{-1}, f^{-1}]f^3 \\ &\equiv [u^{-1}, f^{-1}]^3f^3 \,(\mathrm{mod}\, F_4).\end{aligned}$$

Therefore, if $\gamma \in F' = F_2$ and $\gamma \in \mathrm{Cu}_2(F)$ then the image of γ in F_2/F_3 belongs to $\mathrm{Cu}_1(F_2/F_3)$ and the image of γ in F_2/F_4 belongs to $\mathrm{Cu}_2(F_2/F_4)$.

We show if $\gamma \in F_n$ then $f = uv \in F_n$. Let $\gamma = u^3v^3 \in F_n$ and suppose $uv \notin F_n$. Assume $r \leqslant n$ be the least integer such that $uv \in F_{r-1} - F_r$. Then $(uv)^3 \in F_{r-1} - F_r$, because F_{r-1}/F_r is torsion free. Hence, $u^3v^3 \in F_{r-1} - F_r$ and $\gamma \notin F_r$, a contradiction.

Suppose now that $\gamma \in F_n\,(n \geqslant 2)$. Then $f = uv \in F_n$, hence $[u^{-1}, f^{-1}]$ and $[u^{-1}, f^{-2}]$ belong to F_{n+1} and

$$\gamma = [u^{-1}, f^{-1}]^{u^{-1}}[u^{-1}, f^{-2}]f^3 \equiv f^3 \,(\mathrm{mod}\, F_{n+1}).$$

From the last equation we have

$$\begin{aligned}\gamma &= [u^{-1}, f^{-1}][u^{-1}, f^{-1}, u^{-1}][u^{-1}, f^{-1}]^2[u^{-1}, f^{-1}, f^{-1}]f^3 \\ &\equiv [u^{-1}, f^{-1}]^3 f^3 \,(\mathrm{mod}\, F_{n+2})\end{aligned}$$

and the proof is complete. Now we turn to the proof of Theorem 1. □

Proof of Theorem 1. Let $[x, y, x]^n = A^3B^3$ in F. By Lemma 1 there is a $u \in F_3$, such that $[x, y, x]^n \equiv u^3 \,(\mathrm{mod}\, F_4)$. Since $[x, y, x]$ belongs to a basis of the free group F_2/F_3, it follows that $[x, y, x]^n \equiv u^3$ modulo F_4 iff $3|n$. This completes the proof of Theorem 1. □

Corollary 1. *Let* $\gamma = [c_{i_1}, \ldots, c_{i_k}, c_{i_{k+1}}, c_{i_{k+1}}]$ *be a basic commutator in F. Then* $\mathrm{Cu}(\gamma) = 3$, *in particular if* $\gamma = [x, y, x]$ *then* $\mathrm{Cu}(\gamma) = 3$.

Proof. Set $\omega = [c_{i_1}, \ldots, c_{i_k}]$, so that $\gamma = [[\omega, c_{i_{k+1}}], c_{i_{k+1}}]$. Then by Eq. (1), $\gamma \in \mathrm{Cu}_3(F)$. By Lemma 1, $\gamma^n = A^3B^3$ implies that $\gamma^n \equiv u^3 \,(\mathrm{mod}\, F_{k+3})$, where u is an element of F_{k+2}. Since γ is a basic commutator, it follows that $\gamma^n \equiv u^3 \,(\mathrm{mod}\, F_{k+3})$ iff $3|n$, hence $\mathrm{Cu}(\gamma) \neq 2$. In particular, by Theorem 1 and Eq. (1), $\mathrm{Cu}[x, y, x] = 3$. □

Corollary 2. *With the hypothesis of Corollary* 1, $3 \leqslant \mathrm{Cu}(\gamma^2) \leqslant 4$, *in particular* $3 \leqslant \mathrm{Cu}[x, y, x]^2 \leqslant 4$.

Proof. Since $\gamma^2 = \gamma^{-1}\gamma^3$, Corollary 1 implies that $\mathrm{Cu}(\gamma^2) \leqslant 4$. By Theorem 1 and the proof of Corollary 1, $3 \leqslant \mathrm{Cu}(\gamma^2)$. The proof is complete. □

Proof of Theorem 2. In [1] we proved that every element of W' is a product of at most three commutators in W. Let B be the base group of W and let $T = \langle t \rangle \cong C_\infty$. Then $W = BT$ is the semidirect product of B and T and $W' = [B, T] = \{[a, t],\, a \in B\}$. Similarly, $F_3(W) = [B, T, T] = \{[a, t, t],\, a \in B\}$, hence $\mathrm{Cu}(F_3(W)) \leqslant 3$. According to [1] every element of W' modulo B' is a commutator, and if g is an arbitrary element of B' there are suitable elements say b, c in B such that

$$g = [b^{-1}, t^m][c^{-1}, t^{2m}] = (t^{-m})^{b^{-1}}(t^{-2m})^{c^{-1}t^m}t^{3m}.$$

Without loss of generality, we may assume m is a multiple of 3. Hence, $\mathrm{Cu}(B') \leqslant 3$ and every element of W' can be written as a product of a commutator and three cubes in W. $\square$

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